

Identities in the Superintegrable Chiral Potts Model

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Abstract. We present proofs for a number of identities that are needed to study the superintegrable chiral Potts model in the $Q \neq 0$ sector.

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1. Introduction

The integrable N -state chiral Potts model is defined through a solution of the star-triangle equations parametrized by a Fermat-type curve of genus greater than one [1, 2]. In spite of this difficulty, eigenvalues of the transfer matrix have been obtained using functional equations [3, 4]. There are now detailed results for free energies, interfacial tensions and excitation spectra [5, 6, 7, 8, 9]. As far as the correlation functions are concerned, not much is known beyond the order parameter [10, 11] and some limited partial results on the leading asymptotic behavior of pair correlation functions from conformal field theory and finite-size corrections [12]. Further progress requires the knowledge of eigenvectors of the transfer matrix and form factors.

In order to proceed two recent approaches have been proposed, both starting by studying eigenvectors of the τ_2 model [3, 4]. The first approach [19, 20, 21, 22, 23, 24, 25] begins with the $B_n(\lambda)$ element of the monodromy matrix of the τ_2 model, diagonalizing it for fixed boundary conditions. Using the $B_n(\lambda)$ eigenvectors further results are found for the τ_2 transfer matrix and commuting spin chain Hamiltonians, with the most explicit results for the finite-size $N = 2$ Ising case. The other (our) approach is to start from the superintegrable τ_2 model with periodic boundary conditions in order to find eigenvectors of the N -state chiral Potts model.

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In our two previous papers [13, 14]|| we have expressed the transfer matrices of the superintegrable chiral Potts model for the $Q = 0$ ground state sector—with Q the spin shift quantum number and length L a multiple of the number N of states for each spin—in terms of generators of simple \mathfrak{sl}_2 algebras, which generate the corresponding degenerate eigenspace of $\tau_2(t_q)$. The $\tau_2(t_q)$ used in our papers are directly obtained from the chiral Potts model and are different from those of Nishino and Deguchi [16, 17]. However, as they are related [17, 18], one can easily follow the procedure of [16] to obtain the corresponding loop algebra $L(\mathfrak{sl}_2)$, which can be decomposed into $r = (N-1)L/N$ simple \mathfrak{sl}_2 algebras [14]. Thus we found 2^r chiral Potts eigenvectors for $Q = 0$.

For the $Q \neq 0$ cases, the situation is more difficult, and not much is known about the eigenvectors in the published literature¶, except that some related investigation on the six-vertex model at roots of unity has been done in [15]. As can be seen from e.g. (I.47) in [13], and (45) of the follow-up [18] of this paper, there are different ways to construct the eigenvectors of the degenerate eigenspaces of $\tau_2(t_q)$. It is highly nontrivial, therefore, to determine the generators of the loop algebra $L(\mathfrak{sl}_2)$.

We propose that (II.66) and (II.68) for $Q = 0$ can be generalized to $Q \neq 0$ cases as

$$\langle \Omega | \mathbf{E}_{m,Q}^- = -(\beta_{m,0}^Q / \Lambda_0^Q) \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \langle \{n_j\} | \omega^{\sum_j j n_j} \bar{G}_Q(\{n_j\}, z_{m,Q}), \quad (1)$$

$$\mathbf{E}_{k,Q}^+ | \Omega \rangle = (\beta_{k,0}^Q / \Lambda_0^Q) z_{k,Q} \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \omega^{-\sum_j j n_j} G_Q(\{n_j\}, z_{k,Q}) | \{n_j\} \rangle. \quad (2)$$

where $z_{k,Q}$, G_Q and \bar{G}_Q shall be defined later in the paper. If such a generalization is valid, or, in other words, if the $\mathbf{E}_{m,Q}^\pm$ are indeed the generators of \mathfrak{sl}_2 algebras, then it is necessary that

$$\langle \Omega | \mathbf{E}_{k,Q}^- \mathbf{E}_{m,Q}^+ | \Omega \rangle = -\delta_{k,m} \langle \Omega | \mathbf{H}_k^Q | \Omega \rangle = \delta_{k,m}. \quad (3)$$

In this paper we shall prove this orthogonality relation, which is used in [18] to find generators of an $L(\mathfrak{sl}_2)$ loop algebra for $Q \neq 0$. For $Q = 0$, this proof is not necessary, but can be seen as added confirmation that the decomposition of the loop algebra to \mathfrak{sl}_2 's is correct.

The results of this paper are used in [18] for the case of the superintegrable chiral Potts model with periodic boundary conditions, L a multiple of N and $Q \neq 0$. It should be remarked that the results of this paper also apply to cases with L not a multiple of N , but in the thermodynamic limit $L \rightarrow \infty$ one can ignore these cases. Finally, as the identities in this work may also be related to combinatorics, this paper will be presented with details using a self-contained presentation.

|| All equations in [13] or [14] are denoted here by prefacing I or II to their equation numbers.

¶ Tarasov [26] has written the Bethe Ansatz eigenvectors of the special superintegrable τ_2 transfer matrix with vertical rapidities $p \equiv p'$ for general Q . However, due to the high level of degeneracy of the corresponding eigenvalues, he could not give the related chiral Potts eigenvectors.

2. Drinfeld Polynomials and Other Definitions

In (I.40), we have considered the polynomial

$$\mathcal{Q}(t) = \prod_{j=1}^L \sum_{n_j=0}^{N-1} t^{n_j} = \frac{(1-t^N)^L}{(1-t)^L} = \sum_{m=0}^{L(N-1)} c_m t^m, \quad (4)$$

whose coefficient c_m is the number of L -dimensional vectors $\{n_j\} = \{n_1, \dots, n_L\}$ with elements that are integers $n_j = 0, \dots, N-1$ satisfying the condition $n_1 + \dots + n_L = m$. In this paper, Q denotes a nonnegative integer less than N . The Drinfeld polynomial $P_Q(z)$ is defined in (I.11) and can be expressed in terms of the c_m as

$$\begin{aligned} P_Q(z) &= c_Q + c_{N+Q} z + \dots + c_{m_Q N+Q} z^{m_Q} \\ &= N^{-1} t^{-Q} \sum_{a=0}^{N-1} \omega^{-Qa} \mathcal{Q}(t\omega^a) = N^{-1} t^{-Q} \sum_{a=0}^{N-1} \omega^{-Qa} \frac{(1-t^N)^L}{(1-\omega^a t)^L}, \end{aligned} \quad (5)$$

where

$$m_Q \equiv \lfloor (N-1)L/N - Q/N \rfloor, \quad \omega \equiv e^{2\pi i/N}, \quad z \equiv t^N. \quad (6)$$

We have also defined in (II.60) and (II.61) the L -fold sums

$$K_m(\{n_j\}) = \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ n'_1 + \dots + n'_L = m}} \prod_{j=1}^L \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j N_j}, \quad N_j = \sum_{\ell=1}^{j-1} n_\ell, \quad (7)$$

$$\bar{K}_m(\{n_j\}) = \sum_{\substack{\{0 \leq n'_j \leq N-1\} \\ n'_1 + \dots + n'_L = m}} \prod_{j=1}^L \begin{bmatrix} n_j + n'_j \\ n'_j \end{bmatrix} \omega^{n'_j \bar{N}_j}, \quad \bar{N}_j = \sum_{\ell=j+1}^L n_\ell, \quad (8)$$

where the ω -binomial coefficients [27] are defined by

$$\begin{aligned} [n] &\equiv 1 + \dots + \omega^{n-1}, \quad [n]! \equiv [n][n-1] \dots [1], \\ \begin{bmatrix} n \\ n' \end{bmatrix} &\equiv \frac{[n]!}{[n']! [n-n']!} = \frac{(\omega^{1+n-n'}; \omega)_{n'}}{(\omega; \omega)_{n'}}, \quad (x; \omega)_n \equiv \prod_{j=1}^n (1 - x\omega^{j-1}). \end{aligned} \quad (9)$$

We remove the constraint $n'_1 + \dots + n'_L = m$ in (7) by inserting $\prod t^{n'_j}$ within the L -fold summation and summing over m from 0 to $(N-1)L$. Next we use

$$\begin{bmatrix} n+r \\ r \end{bmatrix} = (-1)^r \omega^{nr+r(r+1)/2} \begin{bmatrix} N-1-n \\ r \end{bmatrix}, \quad (10)$$

$$\sum_{r=0}^s \begin{bmatrix} s \\ r \end{bmatrix} (-1)^r \omega^{r(r-1)/2} x^r = (x; \omega)_s, \quad (11)$$

see (10.2.2c) in [27], to perform each of the L sums, noting $\begin{bmatrix} s \\ r \end{bmatrix} = 0$ for $r > s$. After using

$$\begin{aligned} (x\omega^{s+1}; \omega)_{N-1-s} &= \frac{(1-x^N)}{(x; \omega)_{s+1}} = \frac{(1-x^N)}{(1-x)(x\omega; \omega)_s}, \quad (x; \omega)_s (\omega^s x; \omega)_r = (x; \omega)_{r+s}, \\ \prod_{j=1}^L (t\omega^{1+N_j}; \omega)_{n_j} &= (t; \omega)_{n_1 + \dots + n_L} = (t; \omega)_{N_{L+1}}, \quad N_1 = 0, \quad N_j + n_j = N_{j+1}, \end{aligned} \quad (12)$$

we obtain the resulting generating function for $N_{L+1} = n_1 + \cdots + n_L = kN$ as

$$g(\{n_j\}, t) = (1 - t^N)^{L-k} \prod_{j=1}^L (1 - t\omega^{N_j})^{-1}, \quad (13)$$

which is also given in (II.62). The condition $n'_1 + \cdots + n'_L = m$ in (7) means that $K_m(\{n_j\})$ is the coefficients of t^m in the expansion of $g(\{n_j\}, t)$, i.e.,

$$g(\{n_j\}, t) = \sum_{m=0}^{(N-1)L-kN} K_m(\{n_j\}) t^m. \quad (14)$$

In the same way, we can derive the generating function of $\bar{K}_m(\{n_j\})$. We end up with an equation like (13) where N_j is replaced by $\bar{N}_j = kN - N_{j+1}$, so that

$$\bar{g}(\{n_j\}, t) = \sum_{m=0}^{(N-1)L-kN} \bar{K}_m(\{n_j\}) t^m = g(\{n_j\}, t)^*, \quad (15)$$

which is the complex conjugate of (14) when t is real.

In the special case $n_1 = \cdots = n_L = 0$, we have $k=0$, and it can be seen from (7) that $N_j=0$ for all j . Comparing (13) with (4), we find $g(\{0\}, t) = \bar{g}(\{0\}, t) = \mathcal{Q}(t)$. This also means that $K_m(\{0\}) = \bar{K}_m(\{0\}) = c_m$.

In (II.63) and (II.64) we have defined the polynomials

$$\begin{aligned} G_Q(\{n_j\}, z) &= N^{-1} t^{-Q} \sum_{a=0}^{N-1} \omega^{-Qa} g(\{n_j\}, t\omega^a) = \sum_{m=0}^{m_Q-k} K_{mN+Q}(\{n_j\}) z^m, \\ \bar{G}_Q(\{n_j\}, z) &= N^{-1} t^{-Q} \sum_{a=0}^{N-1} \omega^{-Qa} \bar{g}(\{n_j\}, t\omega^a) = \sum_{m=0}^{m_Q-k} \bar{K}_{mN+Q}(\{n_j\}) z^m. \end{aligned} \quad (16)$$

The polynomials (16) correspond to (14) and (15) in the same way as (5) relates to (4), and for $Q = 0$, they are related to the generators \mathbf{E}_m^\pm of a direct sum of \mathfrak{sl}_2 algebras [14]. We shall show that they satisfy the following orthogonality relation for all Q .

3. Main Theorem and Other Identities

Theorem. *Let the roots of the Drinfeld polynomial $P_Q(z)$ given in (4) be denoted by $z_{j,Q}$, for $j = 1, \dots, m_Q$, i.e.*

$$P_Q(z) = \sum_{m=0}^{m_Q} \Lambda_m^Q z^m = \Lambda_{m_Q}^Q \prod_{j=1}^{m_Q} (z - z_{j,Q}), \quad \Lambda_m^Q \equiv c_{mN+Q}. \quad (17)$$

Then the polynomials (16) satisfy the orthogonality relation

$$\sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \cdots + n_L = N}} \bar{G}_Q(\{n_j\}, z_{i,Q}) G_Q(\{n_j\}, z_{k,Q}) = -B_k \delta_{ik}, \quad (18)$$

where B_k is a constant given by

$$B_k = z_{k,Q} (\Lambda_{m_Q}^Q)^2 \prod_{\ell \neq k} (z_{k,Q} - z_{\ell,Q})^2. \quad (19)$$

When this orthogonality identity holds for any Q , then (II.66)–(II.69) can be generalized to $Q \neq 0$. Originally we used Maple to check it for N and L small, and found that it indeed holds. Here we shall present a proof by first proving a few lemmas.

Definition. Let μ_j and λ_j be integers satisfying $0 \leq \mu_j, \lambda_j \leq N-1$, for $j = 1, \dots, L$, and let

$$a_j = \sum_{\ell=1}^{j-1} \mu_\ell, \quad \bar{a}_j = \sum_{\ell=j+1}^L \mu_\ell, \quad b_j = \sum_{\ell=1}^{j-1} \lambda_\ell, \quad \bar{b}_j = \sum_{\ell=j+1}^L \lambda_\ell. \quad (20)$$

Then $I_m(\{\mu_j\}; \{\lambda_j\})$ is the L -fold sum depending on m , $\{\mu_j\}$ and $\{\lambda_j\}$ and defined by

$$I_m(\{\mu_j\}; \{\lambda_j\}) \equiv \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \begin{bmatrix} \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(a_j - N_j) + n_j \bar{b}_j}, \quad (21)$$

where N_j is defined in (7).

We find that the following identities hold

Lemma 1. For $\mu_1 + \dots + \mu_L = \ell N + Q$, we find

(i) if $\lambda_1 + \dots + \lambda_L = Q$, then

$$I_{\ell N}(\{\mu_j\}; \{\lambda_j\}) = \binom{\ell}{k}; \quad (22)$$

(ii) if $\lambda_1 + \dots + \lambda_L = nN + Q$, then

$$I_N(\{\mu_j\}; \{\lambda_j\}) = (\ell - n) + \bar{I}_N(\{\lambda_j\}; \{\mu_j\}), \quad (23)$$

$$I_{\ell N}(\{\mu_j\}; \{\lambda_j\}) = \bar{I}_{nN}(\{\lambda_j\}; \{\mu_j\}), \quad (24)$$

$$I_{\ell N - N}(\{\mu_j\}; \{\lambda_j\}) = (\ell - n) \bar{I}_{nN}(\{\lambda_j\}; \{\mu_j\}) + \bar{I}_{nN - N}(\{\lambda_j\}; \{\mu_j\}), \quad (25)$$

where

$$\bar{I}_m(\{\lambda_j\}; \{\mu_j\}) = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \begin{bmatrix} \lambda_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \mu_j \\ n_j \end{bmatrix} \omega^{n_j(\bar{b}_j - \bar{N}_j) + n_j a_j}, \quad (26)$$

Proof. First, it is trivial to show $\sum_j n_j N_j = \sum_j n_j \bar{N}_j$, so that for $n_1 + \dots + n_L = m$

$$\sum_{j=1}^L n_j N_j = \frac{1}{2} \sum_{j=1}^L n_j (N_j + \bar{N}_j) = \frac{1}{2} \sum_{j=1}^L n_j (m - n_j) = \frac{1}{2} \left(m^2 - \sum_{j=1}^L n_j^2 \right).$$

Substituting this into (21) and using (9) and (10), we rewrite (21) as

$$(-1)^m \omega^{\frac{1}{2}m^2} I_m(\{\mu_j\}; \{\lambda_j\}) = \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = m}} \prod_{j=1}^L \frac{(\omega^{-\mu_j}; \omega)_{n_j} (\omega^{1+\lambda_j}; \omega)_{n_j}}{(\omega; \omega)_{n_j} (\omega; \omega)_{n_j}} \omega^{n_j(\frac{1}{2} + \mu_j + a_j + \bar{b}_j)}. \quad (27)$$

Its generating function can be written as a product of L sums

$$\begin{aligned} \prod_{j=1}^L J_j(t) &= \sum_{m=0}^{a_{L+1}} (-1)^m \omega^{\frac{1}{2}m^2} I_m(\{\mu_j\}; \{\lambda_j\}) t^m, \quad a_{L+1} = \mu_1 + \cdots + \mu_L; \\ J_j(t) &= \sum_{n_j=0}^{\mu_j} \frac{(\omega^{-\mu_j}; \omega)_{n_j} (\omega^{1+\lambda_j}; \omega)_{n_j}}{(\omega; \omega)_{n_j} (\omega; \omega)_{n_j}} (t\omega^{\frac{1}{2}+\mu_j+a_j+\bar{b}_j})_{n_j} \end{aligned} \quad (28)$$

where the polynomial $J_j(t)$ is a basic hypergeometric function. The transformation formula (10.10.2) of [27] can be rewritten here as

$${}_2\Phi_1 \left[\begin{matrix} q^a, q^b \\ q^c \end{matrix}; q, x \right] = (q^{a+b-c}x; q)_{c-b-a} {}_2\Phi_1 \left[\begin{matrix} q^{c-a}, q^{c-b} \\ q^c \end{matrix}; q, q^{a+b-c}x \right]. \quad (29)$$

Then, setting $q = r\omega$ and letting $r \uparrow 1$ in (29), we obtain

$$\begin{aligned} J_j(t) &= {}_2\Phi_1 \left[\begin{matrix} \omega^{-\mu_j}, \omega^{1+\lambda_j} \\ \omega \end{matrix}; \omega, t\omega^{\frac{1}{2}+\mu_j+a_j+\bar{b}_j} \right] \\ &= (\omega^{\frac{1}{2}+\lambda_j+a_j+\bar{b}_j}t; \omega)_{\mu_j-\lambda_j} {}_2\Phi_1 \left[\begin{matrix} \omega^{1+\mu_j}, \omega^{-\lambda_j} \\ \omega \end{matrix}; \omega, t\omega^{\frac{1}{2}+\lambda_j+a_j+\bar{b}_j} \right]. \end{aligned} \quad (30)$$

From (20), we find that

$$\begin{aligned} a_1 &= 0, \quad a_j + \mu_j = a_{j+1}, \quad a_{L+1} = \ell N + Q, \\ \bar{b}_0 &= \lambda_1 + \cdots + \lambda_L = nN + Q, \quad \bar{b}_j + \lambda_j = \bar{b}_{j-1}, \quad \bar{b}_L = 0, \end{aligned} \quad (31)$$

$$(\lambda_{j+1} + a_{j+1} + \bar{b}_{j+1}) - (\lambda_j + a_j + \bar{b}_j) = \mu_j - \lambda_j. \quad (32)$$

Using (31), (32) and the second identity in (12), we find

$$\begin{aligned} \prod_{j=1}^L (\omega^{\frac{1}{2}+\lambda_j+a_j+\bar{b}_j}t; \omega)_{\mu_j-\lambda_j} &= (\omega^{\frac{1}{2}+\bar{b}_0}t; \omega)_{a_{L+1}-\bar{b}_0} \\ &= (\omega^{\frac{1}{2}+\bar{b}_0}t; \omega)_{\ell N - nN} = (1 + t^N)^{\ell-n}. \end{aligned} \quad (33)$$

The generating function of the sums (26) is given by

$$\begin{aligned} \prod_{j=1}^L \bar{J}_j(t) &= \sum_{m=0}^{\bar{b}_0} (-1)^m \omega^{\frac{1}{2}m^2} \bar{I}_m(\{\lambda_j\}; \{\mu_j\}) t^m, \\ \bar{J}_j(t) &= {}_2\Phi_1 \left[\begin{matrix} \omega^{1+\mu_j}, \omega^{-\lambda_j} \\ \omega \end{matrix}; \omega, t\omega^{\frac{1}{2}+\lambda_j+a_j+\bar{b}_j} \right], \end{aligned} \quad (34)$$

in analogy with (28). Then using (30), (33) and (34), we find

$$\prod_{j=1}^L J_j(t) = (1 + t^N)^{\ell-n} \prod_{j=1}^L \bar{J}_j(t). \quad (35)$$

By equating the coefficients of t^m on both sides of (35), we relate the sums I_m and \bar{I}_m . Particularly, for $\bar{b}_0 = Q$, $n = 0$, we can see from (34) that the generating function of \bar{I}_m is a polynomial of order Q and that the coefficient of t^{kN} can only come from the factor in front. Furthermore, as $(-1)^{kN} \omega^{\frac{1}{2}(kN)^2} = 1$, this proves (22). For $n \neq 0$, by equating

the coefficients of t^N in (35), we prove (23). Finally, (24) and (25) are proved equating the coefficients of $t^{\ell N}$ and $t^{\ell N - N}$ in (35). \square

As a consequence of Lemma 1, we can prove the following identities:

Lemma 2. *The sums $K_m(\{n_j\})$ and $\bar{K}_m(\{n_j\})$ defined in (7) and (8) satisfy the following relations*

$$(i) \quad \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = kN}} \bar{K}_{\ell N + Q}(\{n_j\}) K_Q(\{n_j\}) = \binom{\ell + k}{k} \Lambda_0^Q \Lambda_{\ell+k}^Q, \quad \Lambda_m^Q = c_{mN+Q}; \quad (36)$$

$$(ii) \quad \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{K}_{\ell N + Q}(\{n_j\}) K_{mN+Q}(\{n_j\}) = \sum_{n=0}^m (\ell + 1 + 2n - m) \Lambda_{m-n}^Q \Lambda_{\ell+1+n}^Q. \quad (37)$$

Proof. Denote

$$\Theta_{\ell, m, k} \equiv \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = kN}} \bar{K}_{\ell N + Q}(\{n_j\}) K_{mN+Q}(\{n_j\}). \quad (38)$$

Using (7) and (8), we find

$$\Theta_{\ell, m, k} = \sum_{\substack{\{0 \leq \lambda_j, n_j, n'_j \leq N-1\} \\ n_1 + \dots + n_L = kN}} \prod_{j=1}^L \begin{bmatrix} n'_j + n_j \\ n_j \end{bmatrix} \begin{bmatrix} n_j + \lambda_j \\ n_j \end{bmatrix} \omega^{n_j(N'_j + \bar{b}_j)}, \quad (39)$$

where

$$\begin{aligned} N'_j &= \sum_{i < j} n'_i, & N'_{L+1} &= n'_1 + \dots + n'_L = \ell N + Q, \\ \bar{b}_j &= \sum_{i > j} \lambda_i, & \bar{b}_0 &= \lambda_1 + \dots + \lambda_L = mN + Q. \end{aligned} \quad (40)$$

In order to apply (21) we change the L summation variables n'_j to $\mu_j = n_j + n'_j$. Since $n_j < N$, we find that the summand in (39) is nonzero if and only if $\mu_j < N$. Also, $a_{L+1} = \mu_1 + \dots + \mu_L = (\ell + k)N + Q$. Thus we may rewrite (39) as

$$\Theta_{\ell, m, k} = \sum_{\substack{\{0 \leq \mu_j \leq N-1\} \\ \mu_1 + \dots + \mu_L = (\ell+k)N+Q}} \sum_{\substack{\{0 \leq \lambda_j \leq N-1\} \\ \lambda_1 + \dots + \lambda_L = mN+Q}} I_{kN}(\{\mu_j\}; \{\lambda_j\}). \quad (41)$$

Now we let $m = 0$ and using (22) with ℓ replaced by $\ell + k$ we find

$$\Theta_{\ell, 0, k} = \sum_{\substack{\{0 \leq \mu_j \leq N-1\} \\ \mu_1 + \dots + \mu_L = (\ell+k)N+Q}} \sum_{\substack{\{0 \leq \lambda_j \leq N-1\} \\ \lambda_1 + \dots + \lambda_L = Q}} \binom{\ell + k}{k} = \binom{\ell + k}{k} \Lambda_{\ell+k}^Q \Lambda_0^Q, \quad (42)$$

where $K_{mN+Q}(\{0\}) = c_{mN+Q} = \Lambda_m^Q$ is used. This proves (36).

Choosing $k = 1$ in (41) and using (23) with ℓ replaced by $\ell + 1$, we find

$$\begin{aligned}\Theta_{\ell,m,1} &= (\ell + 1 - m)\Lambda_{\ell+1}^Q \Lambda_m^Q + \sum_{\substack{\{0 \leq \mu_j \leq N-1\} \\ \mu_1 + \dots + \mu_L = (\ell+1)N+Q}} \sum_{\substack{\{0 \leq \lambda_j \leq N-1\} \\ \lambda_1 + \dots + \lambda_L = mN+Q}} \bar{I}_N(\{\lambda_j\}, \{\mu_j\}) \\ &= (\ell + 1 - m)\Lambda_{\ell+1}^Q \Lambda_m^Q + \Theta_{\ell+1,m-1,1},\end{aligned}\quad (43)$$

where we have used (39) with $k = 1$ after first replacing λ_j by $\lambda_j + n_j$ and μ_j by n'_j in (26). Note that summands with $\lambda_j < n_j$ in (26) or with $\lambda_j + n_j > N$ in (39) vanish. We next apply (43) to $\Theta_{\ell+1,m-1,1}$ and repeat the process until arriving at $\Theta_{\ell+m,0,1}$. Then we can use (36) with $k = 1$ to obtain (37). \square

Remark. Substituting $n = m - j$ we may rewrite (37) as

$$\Theta_{\ell,m,1} = \sum_{j=0}^m (\ell + 1 + m - 2j) \Lambda_j^Q \Lambda_{\ell+1+m-j}^Q. \quad (44)$$

Now we are ready to prove the orthogonality theorem.

Proof. As in (II.72) we introduce the polynomial

$$\mathbf{h}_k^Q(z) \equiv \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{G}_Q(\{n_j\}, z_{k,Q}) G_Q(\{n_j\}, z). \quad (45)$$

Substituting (16) with $k = 1$ into this equation, we find

$$\mathbf{h}_k^Q(z_{i,Q}) = \sum_{\ell=0}^{m_Q-1} \sum_{m=0}^{m_Q-1} z_{k,Q}^\ell z_{i,Q}^m \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} \bar{K}_{\ell N+Q}(\{n_j\}) K_{mN+Q}(\{n_j\}). \quad (46)$$

Now we use Lemma 2(ii) or (44) to write

$$\mathbf{h}_k^Q(z_{i,Q}) = \sum_{\ell=0}^{m_Q-1} \sum_{m=0}^{m_Q-1} z_{k,Q}^\ell z_{i,Q}^m \sum_{j=0}^m (\ell + m + 1 - 2j) \Lambda_j^Q \Lambda_{\ell+m+1-j}^Q. \quad (47)$$

Interchanging the order of summation over m and j , and then letting $m = n + j - \ell - 1$, we find

$$\mathbf{h}_k^Q(z_{i,Q}) = \sum_{\ell=0}^{m_Q-1} \sum_{j=0}^{m_Q-1} \sum_{n=\ell+1}^{m_Q+\ell-j} z_{k,Q}^\ell z_{i,Q}^{n+j-\ell-1} (n-j) \Lambda_j^Q \Lambda_n^Q. \quad (48)$$

Since $\Lambda_n^Q = 0$ for $n > m_Q$, the intervals of summation may be extended to $0 \leq \ell, j \leq m_Q$.

We split the summation over n into three parts as⁺

$$\mathbf{h}_k^Q(z_{i,Q}) = \sum_{\ell=0}^{m_Q} \left(\frac{z_{k,Q}}{z_{i,Q}} \right)^\ell \sum_{j=0}^{m_Q} \left[\sum_{n=0}^{m_Q} - \sum_{n=0}^{\ell} - \sum_{n=m_Q+\ell-j+1}^{m_Q} \right] (n-j) \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1}. \quad (49)$$

⁺ Note that it follows from (4) and (5) that none of the $z_{i,Q}$ vanishes.

The contribution due to the first part is identically zero, as it is antisymmetric under the interchange of the summation variables j and n . The terms with $j \leq \ell$ of the third part are zero, as $\Lambda_n^Q = 0$ for $n > m_Q$, leaving the nontrivial terms

$$\begin{aligned} \sum_{j=\ell+1}^{m_Q} \sum_{n=m_Q+\ell-j+1}^{m_Q} (n-j) \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} &= \sum_{n=\ell+1}^{m_Q} \sum_{j=m_Q+\ell-n+1}^{m_Q} (n-j) \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} \\ &= \sum_{j=\ell+1}^{m_Q} \sum_{n=m_Q+\ell-j+1}^{m_Q} (j-n) \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} = 0. \end{aligned} \quad (50)$$

Here the first equality is obtained by interchanging the order of the summations, the second by interchanging n and j . This shows that the third part is also identically zero.

We split the second part into two pieces with summands proportional to n and to j . In the first piece we can perform the sum over j , which by (17) yields a factor $P_Q(z_{i,Q})$. This shows that the only nonvanishing contribution comes from the second piece, or

$$h_k^Q(z_{i,Q}) = \sum_{\ell=0}^{m_Q} \left(\frac{z_{k,Q}}{z_{i,Q}} \right)^\ell \sum_{j=0}^{m_Q} \sum_{n=0}^{\ell} j \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} = \sum_{j=0}^{m_Q} \sum_{n=0}^{m_Q} j \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} \sum_{\ell=n}^{m_Q} \left(\frac{z_{k,Q}}{z_{i,Q}} \right)^\ell. \quad (51)$$

We first consider the case $z_{k,Q} \neq z_{i,Q}$, so that the last sum can be evaluated to give

$$h_k^Q(z_{i,Q}) = \frac{1}{1 - (z_{k,Q}/z_{i,Q})} \sum_{j=0}^{m_Q} \sum_{n=0}^{m_Q} j \Lambda_j^Q \Lambda_n^Q z_{i,Q}^{n+j-1} \left[\left(\frac{z_{k,Q}}{z_{i,Q}} \right)^n - \left(\frac{z_{k,Q}}{z_{i,Q}} \right)^{m_Q+1} \right] = 0, \quad (52)$$

as seen again from (17) after summing over n . Finally, we consider the case $z_{k,Q} = z_{i,Q}$, so that the last sum is $(m_Q - n + 1)$ leading to

$$h_k^Q(z_{k,Q}) = \sum_{j=0}^{m_Q} \sum_{n=0}^{m_Q} j \Lambda_j^Q \Lambda_n^Q z_{k,Q}^{n+j-1} (m_Q - n + 1) = -z_{k,Q} \left[\sum_{j=0}^{m_Q} j \Lambda_j^Q z_{k,Q}^{j-1} \right]^2, \quad (53)$$

after using (17) to show that the $(m_Q + 1)$ terms do not contribute. We use (17) once more to show

$$P'_Q(z) = \sum_{j=0}^{m_Q} j \Lambda_j^Q z^{j-1} = \Lambda_{m_Q}^Q \left(\prod_{j \neq k} (z - z_{j,Q}) + (z - z_{k,Q}) \frac{d}{dz} \prod_{j \neq k} (z - z_{j,Q}) \right). \quad (54)$$

Consequently, we have

$$\sum_{j=0}^{m_Q} j \Lambda_j^Q z_{k,Q}^{j-1} = \Lambda_{m_Q}^Q \prod_{j \neq k} (z_{k,Q} - z_{j,Q}), \quad h_k^Q(z_{k,Q}) = z_{k,Q} (\Lambda_{m_Q}^Q)^2 \prod_{j \neq k} (z_{k,Q} - z_{j,Q})^2. \quad (55)$$

This is identical to B_k in (19). Thus we have finished the proof. \square

We note that we have not used the assumption that all roots $z_{i,Q}$ for given L and Q are distinct. The theorem holds also, if there is a multiple root $z_{k,Q}$, for which $B_k = 0$. However, there is substantial numerical evidence that the roots may all be distinct and we shall assume this for the following corollary of the theorem.

The degree of polynomial (45) is $m_Q - 1$, as can be seen from (16) with $k = 1$, whereas polynomial (II.10) [14, 28]

$$f_k^Q(z) = \prod_{\ell \neq k} \frac{z - z_{\ell,Q}}{z_{k,Q} - z_{\ell,Q}} = \sum_{n=0}^{m_Q-1} \beta_{k,n}^Q z^n, \quad f_k^Q(z_{j,Q}) = \delta_{j,k} \quad (56)$$

is of the same degree and has the same roots. These polynomials must be equal except for some multiplicative constant. Using the orthogonality theorem, we find the identity

$$\mathbf{h}_k^Q(z) = -B_k f_k^Q(z) = -z_{k,Q} (\Lambda_{m_Q}^Q)^2 \prod_{\ell \neq k} [(z_{k,Q} - z_{\ell,Q})(z - z_{\ell,Q})], \quad (57)$$

valid when all roots are distinct.

Finally, we can also introduce the polynomials

$$\bar{\mathbf{h}}_k^Q(z) \equiv \sum_{\substack{\{0 \leq n_j \leq N-1\} \\ n_1 + \dots + n_L = N}} G_Q(\{n_j\}, z_{k,Q}) \bar{G}_Q(\{n_j\}, z). \quad (58)$$

Since the roots $z_{k,Q}$ are real, $\bar{\mathbf{h}}_k^Q(z)$ is the complex conjugate of $\mathbf{h}_k^Q(z)$ for real z . But by (57) we see that $\mathbf{h}_k^Q(z)$ is real in that case, so that

$$\bar{\mathbf{h}}_k^Q(z) = \mathbf{h}_k^Q(z), \quad \text{if } z \in \mathbb{R}. \quad (59)$$

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